Monoidal Theories and Graph Rewriting in Quantum Computing

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- Most importantly, we have a graphical notation
- Applications to quantum states and processes motivates the study of graphical, monoidal theories in categories like FdHilb
- To do graphical proofs (esp. automatic ones), we need the correct notion of graph rewriting, as well as sound, efficient algorithms

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- Mixed state quantum mechanics has generalisations of the above. We won't talk about that.

Entanglement and the Tensor

- For our purposes, take $\otimes$ to be the Kronecker product:

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- For Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we can construct $\mathcal{H}_1 \otimes \mathcal{H}_2 = \operatorname{span}\{v \otimes u : v \in \mathcal{H}_1, u \in \mathcal{H}_2\}$.
  - $\dim(\mathcal{H}_1 \otimes \mathcal{H}_2) = \dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2$

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- The Hilbert space $\mathcal{Q} := \mathbb{C}^2$ is called the space of *qubits*.
- We write the standard basis of $\mathcal{Q}$ in "ket" notation, as $|0\rangle, |1\rangle$. Also, $|ij\rangle$ is shorthand for $|i\rangle \otimes |j\rangle$.

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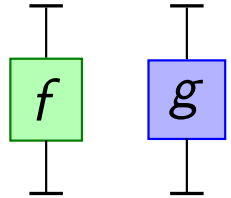
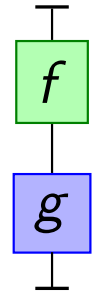
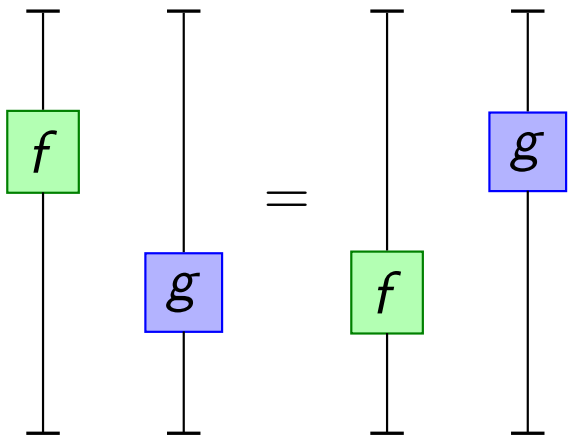
- Other examples include Rel, nCob, *Span*($\mathcal{C}$), *Csp*($\mathcal{C}$), group representation categories, projective spaces, ...

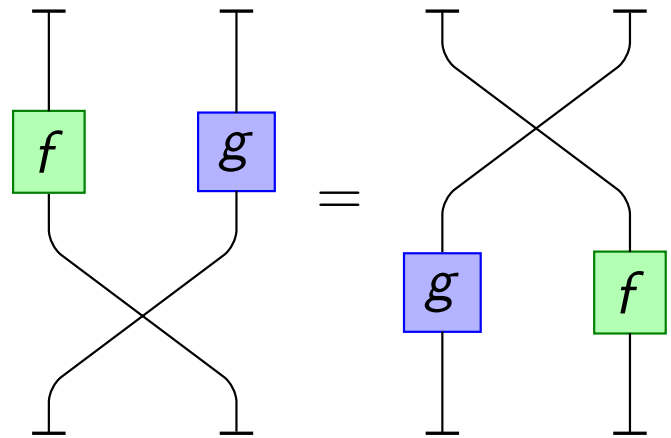
Graphical Representation

$f \otimes g :=$ 

$g \circ f :=$ 

- We can express the bifunctoriality of $\otimes$ and the naturality of σ as follows:



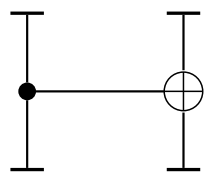


Quantum Circuits

- Any †-SMC provides a notion of *unitarity* ($U^\dagger U = 1$ and $UU^\dagger = 1$)
- We can think of unitary maps as the quantum analogy to reversible logic gates
- Lets look at one motivating example:

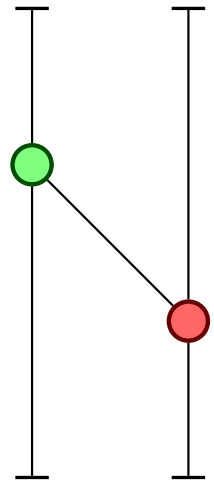
$$\mathrm{CNOT} :: |00\rangle \to |00\rangle, |01\rangle \to |01\rangle, |10\rangle \to |11\rangle, |11\rangle \to |10\rangle$$

- Usually drawn like this:



More Primitive

- Lets break CNOT into its primitive, (co)algebraic components



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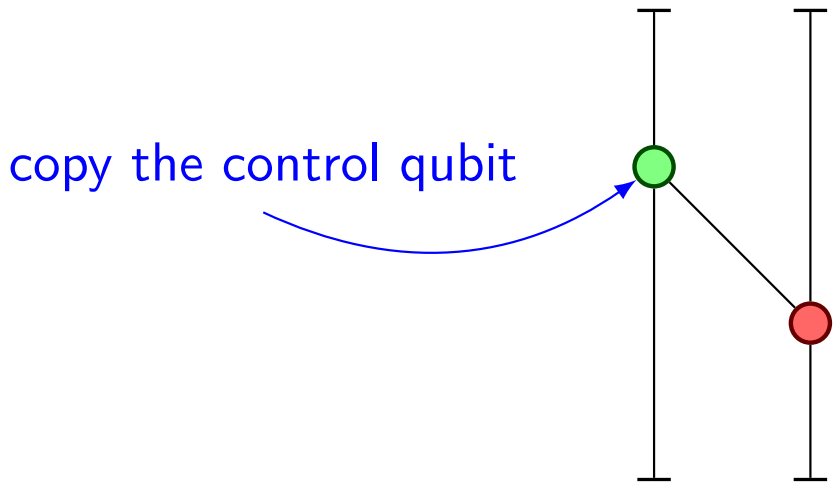

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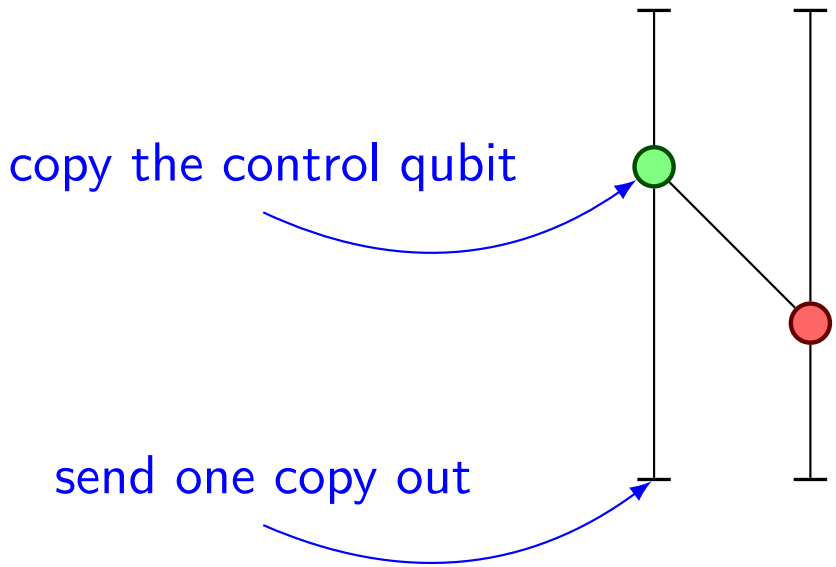

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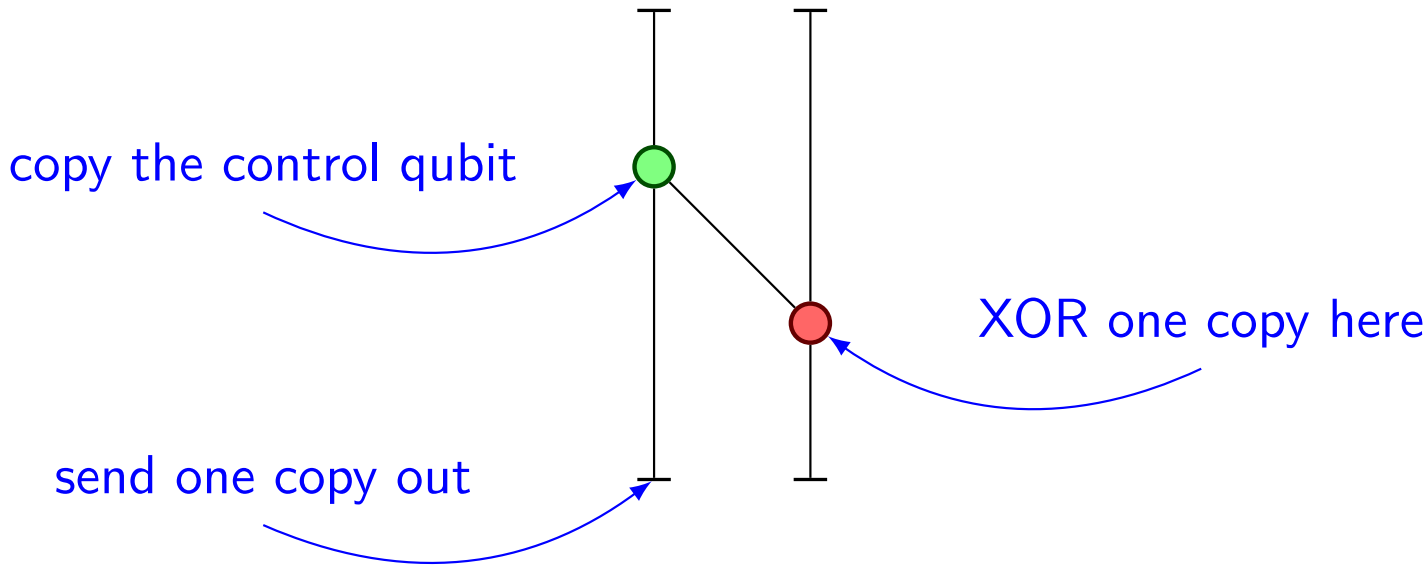
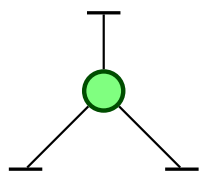


Classical Structures

- A chosen basis is like some classical data embedded in the system.
- What can we do with classical data?
  - Copy and delete!

$$\delta_Z : \mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q} :: |i\rangle \mapsto |ii\rangle \qquad \epsilon_Z : \mathcal{Q} \to \mathbb{C} :: |i\rangle \mapsto 1$$

- Graphically:

$\delta_Z :=$  $\qquad \epsilon_Z :=$ 

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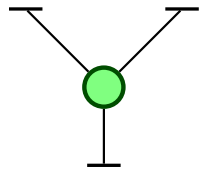
$\delta_Z^\dagger :=$ 

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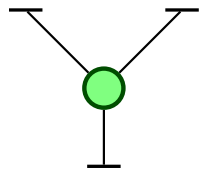
$\delta_Z^\dagger :=$  $\epsilon_Z^\dagger :=$ 

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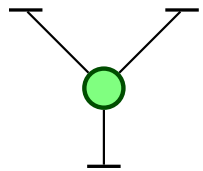
$\delta_Z^\dagger :=$  $\epsilon_Z^\dagger :=$ 

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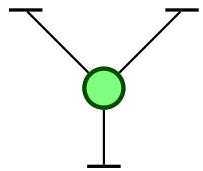
$\delta_Z^\dagger :=$  $\epsilon_Z^\dagger :=$ 

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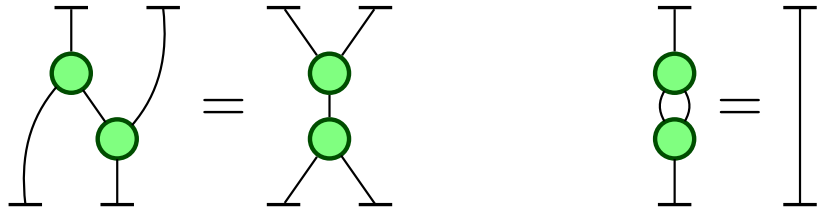
- Therefore, we say $(\delta_Z^\dagger, \epsilon_Z^\dagger)$ is an *internal commutative monoid in FdHilb*
- Likewise, (δ_Z, ϵ_Z) is an *internal cocommutative comonoid*

Spiders

- (δ_Z, ϵ_Z) interacts with $(\delta_Z^\dagger, \epsilon_Z^\dagger)$ in the following way:



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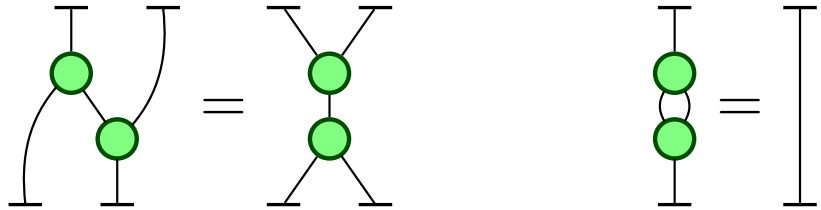
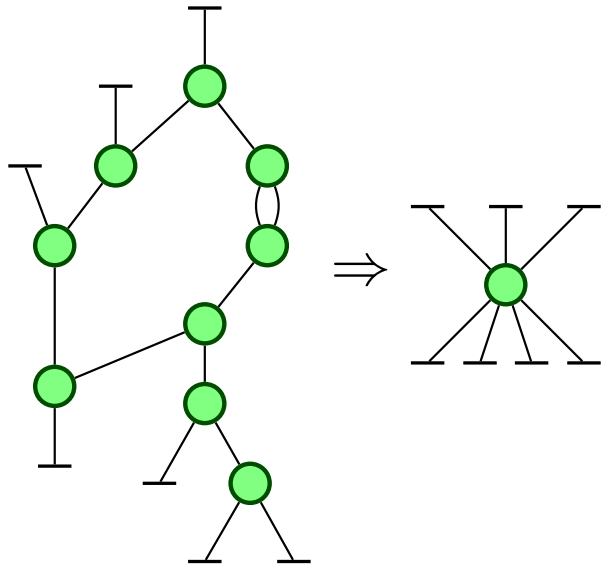


- Terms of these things are uniquely determined by number of inputs and outputs. As a result, we write connected graphs:



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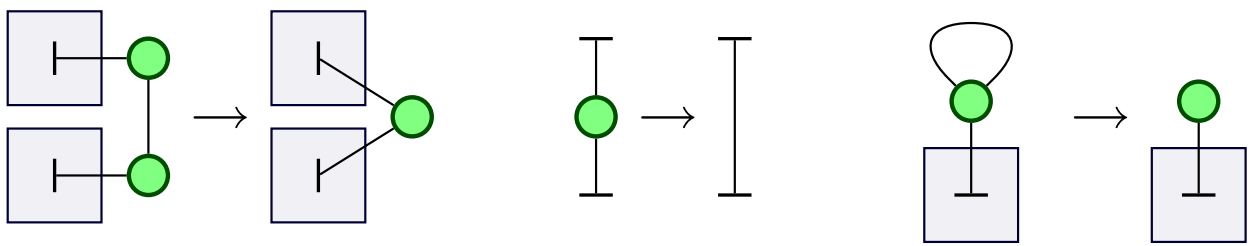

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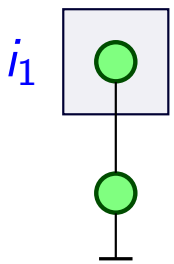

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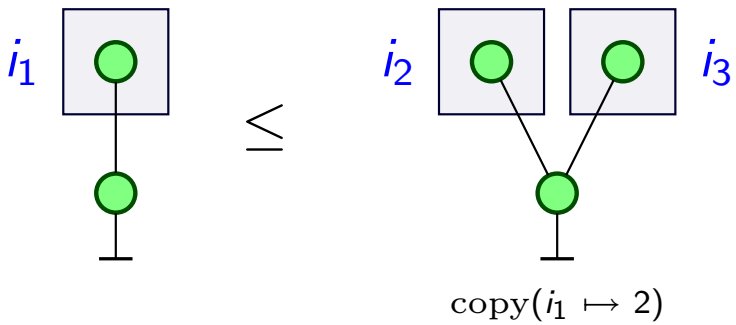

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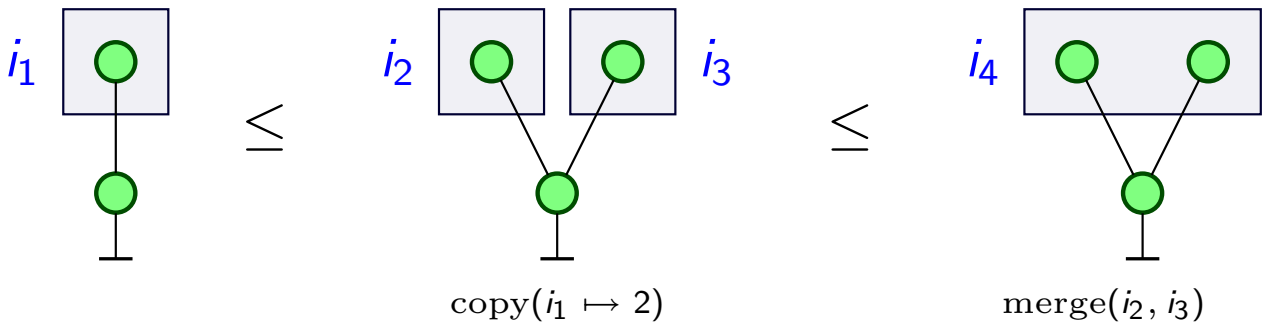

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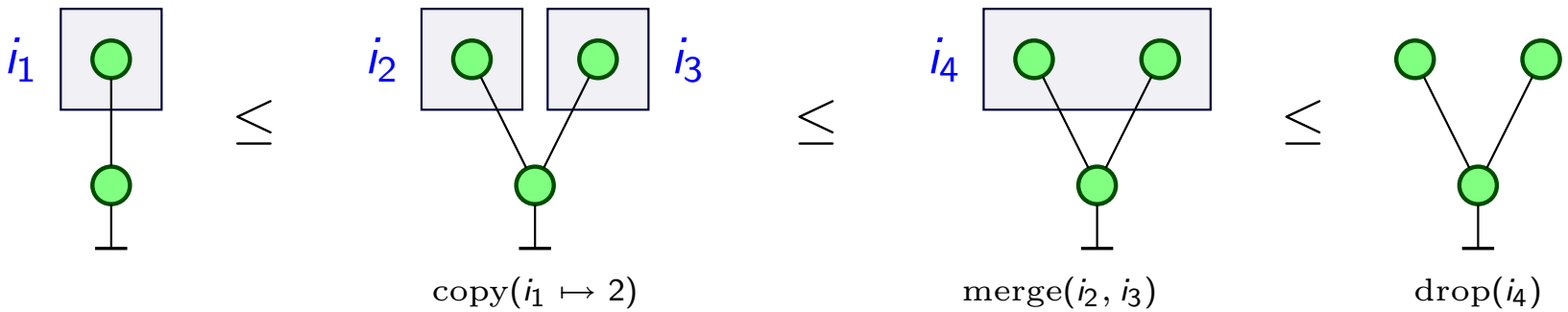

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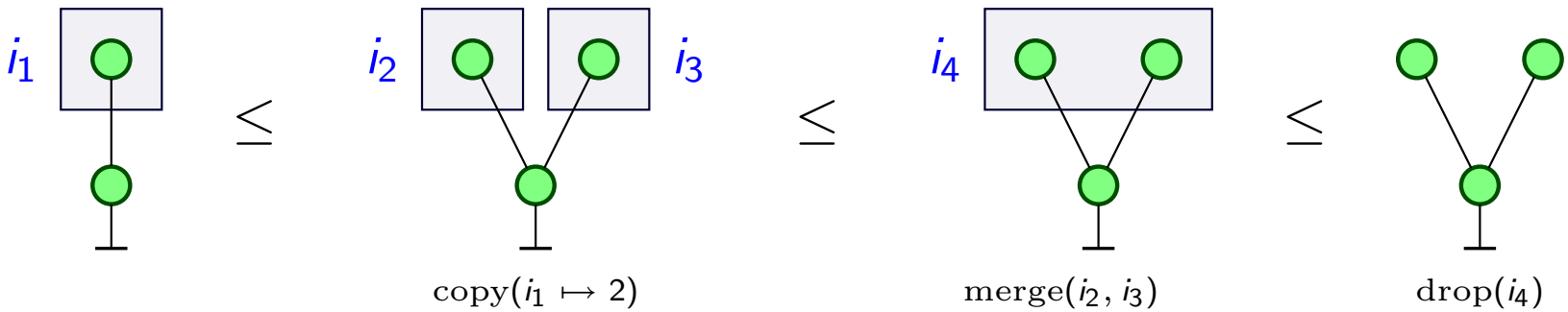

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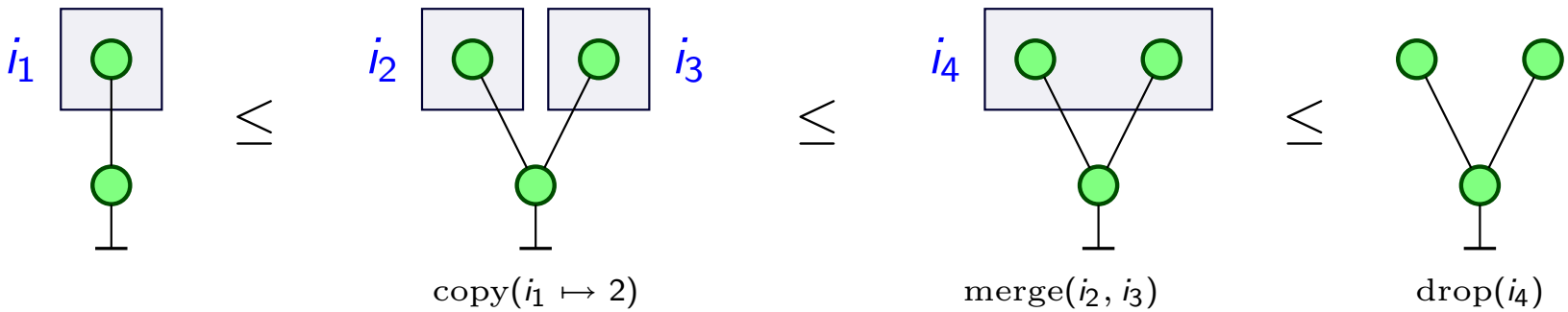

- Naive rewriting algorithm is to expand all combinations of !-boxes, bounded by the size of the target
- Better algorithm expands lazily during matching

Another colour

- Let (δ_X, ϵ_X) copy and delete this basis:

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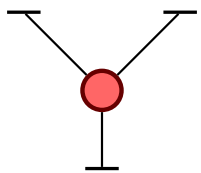
- These two bases are called *mutually unbiased*. They are eigenvectors of *maximally non-commuting observables*. Think position and momentum
- We can depict (δ_X, ϵ_X) with red dots, and note $\delta_X^\dagger$ is the XOR



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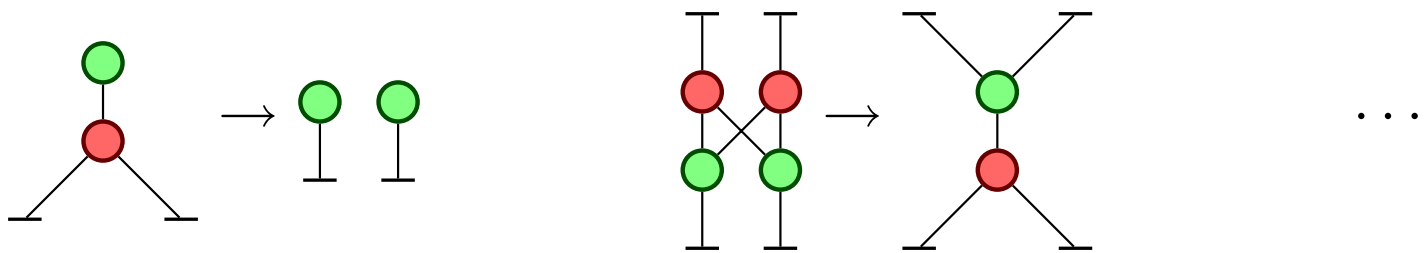
- This yields some more basic rewrites.



Interactions, and a Rewrite Theory

- Red and green are both special frobenius algebras, but they also interact
- For instance, the quadruple $(\delta_Z, \epsilon_Z, \delta_X^\dagger, \epsilon_X^\dagger)$ forms another kind of monoidal gadget, called a *Hopf algebra*
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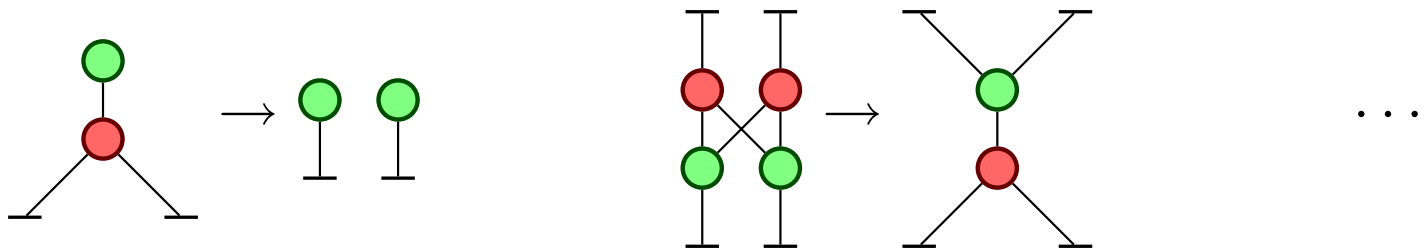
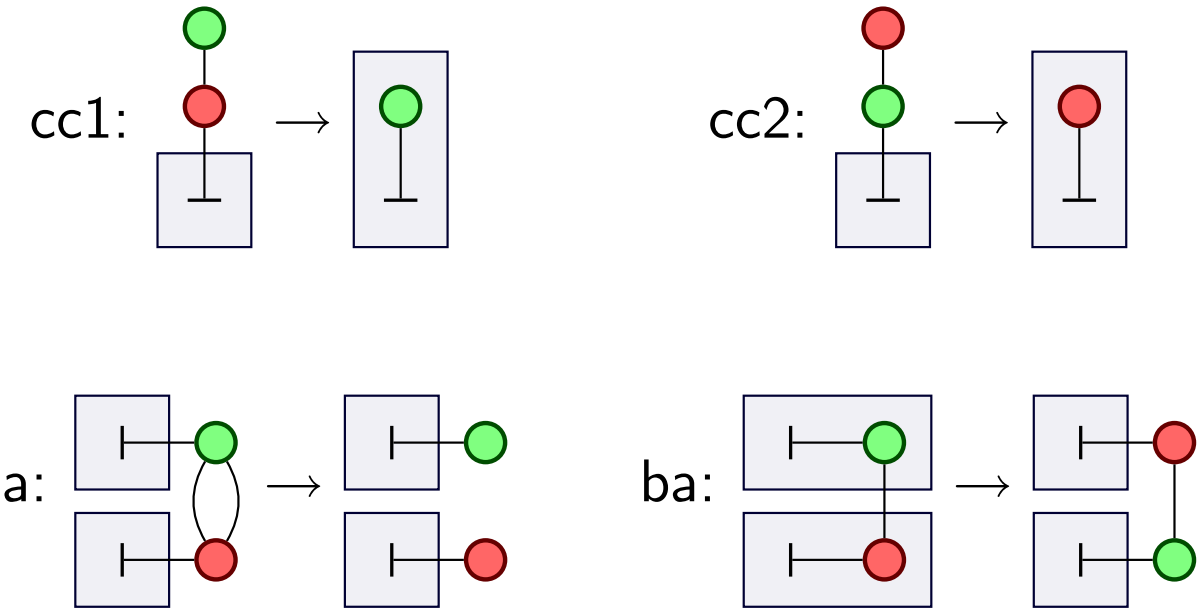


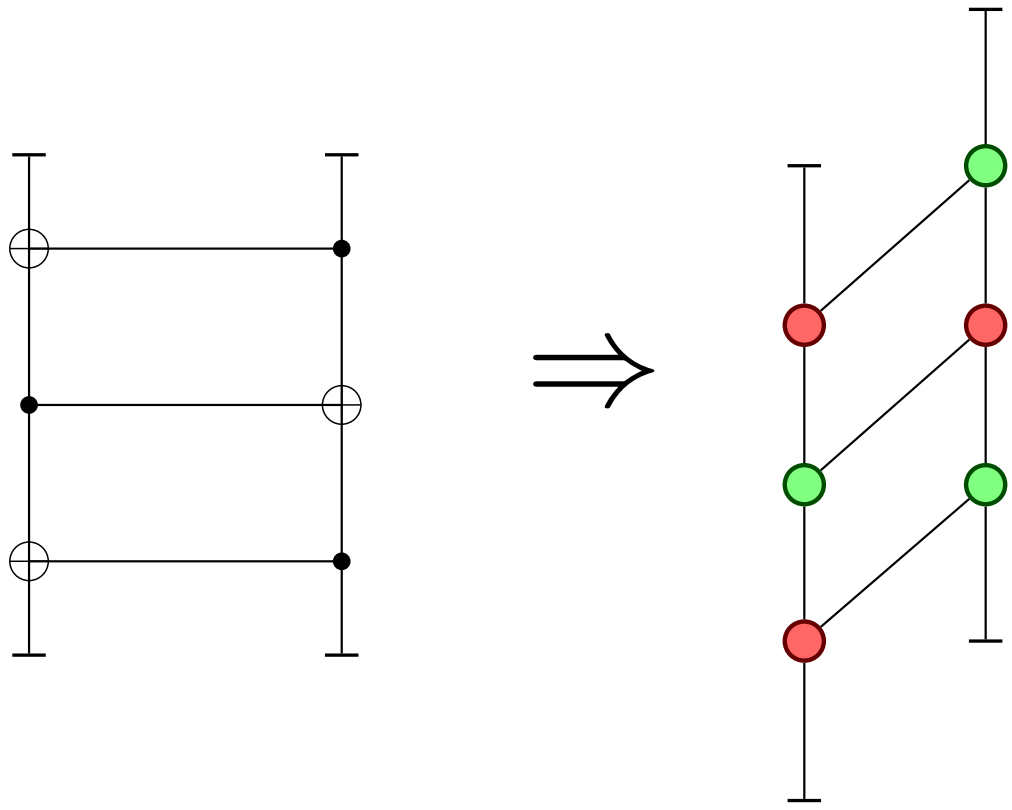
- We can apply pattern rewrites to pattern graphs, so we get...

Interactions, and a Rewrite Theory

- **Derived** pattern rewrites:



Doin' the Swap



Conclusions

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- These yield spider theorems, necessitate pattern rewriting
- We have a tool for doing automatic rewriting, already being used in study of entangled states (numerically and with rewrites)
- Future work: polish the algorithm, more theories, automatic critical pair analysis

Thanks!

- This is joint work with
  - Bob Coecke
    http://www.comlab.ox.ac.uk/people/bob.coecke/
  - Ross Duncan
    http://www.comlab.ox.ac.uk/people/ross.duncan/
  - Lucas Dixon
    http://homepages.inf.ed.ac.uk/ldixon/
- Check it out at
  - http://dream.inf.ed.ac.uk/projects/quantomatic/