Graphical Reasoning in Symmetric Monoidal Categories for Quantum Information

Lucas Dixon, University of Edinburgh
(Joint work Ross Duncan and Aleks Kissinger)

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Graphs: ubiquitous and beautiful

- Abstracts over detail to highlight important features
  - Use spatial layout for names and edges for binding

- Visual intuition: (dis)connectedness, paths, and inclusion encode meaning

- A rich and widely used notation: (formally and informally)
  - Specification/Design: UML
  - Modelling/Simulation: Biological Networks, Quantum Information
  - Engineering/Optimisation: Finite State Machines
  - Proof: Quantum Mechanics and Information

- Applications in... Biology; Physics; Engineering; Computer Science

Graphs: UML

Graphs: Biological Interactions

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Graphs: State Machines

Graphs: circuits, electrical and quantum

Graphs: Feynman Diagrams
Expressivity breeds complexity!

- **Visualise**: Human understanding is modulo graph-layout.

- Families of similar graphs? (e.g. common repeated structures)

- Difficult to reason about: Composition? Equivalences? Induction/recursion principles? Many generic algorithms, such as depth-first search/spanning trees, do not respect the domain’s semantics.

- Dynamics? Interesting systems are not static, how do graphs change? proof, evaluation, evolution...

Complexity breeds computer science?

- **Visualise**: use other people’s solutions...

- Families: formal account of ellipses notation

\[
\begin{array}{c}
\vdots \vdots \\
\vdots \vdots \\
\end{array}
\end{equation}

- **Formalism** that allows computer aided manipulation

- Dynamics captures by graph rewriting (evolution, evaluation)

- **Need**: practical tools to represent and reason with graphs

Tensor (⊗) Graphs

- Graphs where **Vertices** are processes/functions/relations and **Edges** are states/objects and paths describe change/flow of information.

\[
f : A \rightarrow B
\]

\[
I : A \rightarrow A \text{ (identity)}
\]

\[
h : (A \otimes B) \rightarrow C
\]

- These are graphical presentations of monoidal categories.

- **Coherence results**: prove equivalence between categories and graphs
  (Mac Lane, Kelly and Laplaza, Joyal and Street, Selinger)
Lots of kinds of monoidal graphs

Symmetric Monoidal Categories

Bifunctoriality of $\otimes$

$$(f \otimes I) \circ (I \otimes g) = (I \otimes f) \circ (g \otimes I)$$

Symmetry of braiding ($\sigma$)

$$\sigma \circ \sigma = I$$

Sliding Boxes: tensor/\sigma commutativity

$$(f \otimes g) \circ \sigma = \sigma \circ (g \otimes f)$$

Example: Boolean Circuit Graphical Equations

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Expressiveness

- Symmetries suggest changing representation:

$$\begin{array}{c}
\begin{array}{c}
\text{And} \quad \text{Not} \quad \text{Bool} \quad \text{Ignore} \quad \text{Copy} \quad \text{Id}
\end{array}
\end{array}$$

- A formal (and finitely expressible) ellipses notation?

- Traditional graph transformation machinery lacks: graphical derived equations, interfaces for graphs, graphical ellipsis notation.

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Formalism and Implementation

- Need a formal account of graphs, half-edges, tensor, plugging, ellipses notation, matching, and finally rewriting.

Graph: \((s : E \mapsto V, t : E \mapsto V)\)

\[
(s := \{e_1 \mapsto v_1, e_2 \mapsto v_1\}, t := \{e_1 \mapsto v_1, e_2 \mapsto v_1\})
\]

(not a binary relation!)

- Exterior vertices (‘half’-edges) := a subset of the vertices; These define the graph’s interface; can have many incident edges!

- Tensor := disjoint union

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Matching

Cut-out the source graph from the target graph

e.g. cut edge \(x-y\), and cut vertex \(z\):

\((\not\leq)\)

Unify edges to exterior, respect the vertices, e.g:

\(\approx\) a minimal graph w.r.t. shared joining graph

(Defined as a pushout)

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Plugging

The interpretation of !-box graph is the family of ground expansions

\[
(A : \leq \text{merge } \{i_0, i_1\} \quad B : \leq \text{drop } \{i_2\} \quad C : \leq \text{drop } \{i_2\} \quad D : \leq \text{drop } \{i_3\})
\]

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Properties

Plugging and rewriting commute:

\[ A \circ B = A' \circ B \]

Adequacy:

\[ A \rightarrow B = A' \rightarrow B \]

Matching forms a partial order.

Quantum Information: why is it important

- **Quantum Money**: unforgeable (still blue-sky)
- **Quantum Computation**: speedup? polynomial-time factorisation (significant experimental work under way)
- **Quantum Protocols**: Secure communication - built-in physically (Have working systems, claims to be in mobile devices in a few years)
- **Complexity Results**: Alternative formulations of complexity classes, e.g. quantum accounts of PSPACE (Jain, Ji, Upadhyay, Watrous 2009) and EXP (Kobayashi, Matsumoto, 2003)

Quantum Information

- **Quantum computation as circuit-like graphs** (Abramsky, Duncan, Coecke, ...)
- Easy to understand and manipulate.
- Certain properties have a natural graphical representation (e.g. disjointness in graph \( \Rightarrow \) separable state)
- Abstract algebra of graphs has other applications
- Hard to reason with manually \( \Rightarrow \) develop tool support: Quantomatic

Quantumness: Complementary Observables

- [Image of quantumness concept]
Summary

- Graphical presentations of monoidal categories as tensor-graphs
- Formalised and implemented tensor-graphs with interfaces, some ellipses notation, and compositional matching/rewriting
- Supports reasoning about quantum information
- Graphical-equations as a tool for representation and reasoning
- Closely related formalisms: Bi-graphs (+hierarchical nodes define spaces, -no ellipses notation); Graph transformations (not for tensor graphs)
- Thanks!