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# Graph Rewriting for Classical Structures

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**Complementary classical structures** correspond concretely to copy and delete operations over a pair of mutually unbiased bases. When acting as the generators of a †-symmetric monoidal category, they provide a powerful computational primitive, with a simple graphical representation. We have shown that the induced graph rewrite system is well-behaved and lends itself to automation.

## concretely

A vector $\varphi \in \mathcal{H}$ is **unbiased** w.r.t. to a basis $\{\psi_i\}_i$ if $\langle\varphi|\psi_i\rangle = \sqrt{dim\,\mathcal{H}}$ for all $i$.

Two bases $B_1$, $B_2$ are **complementary** if all of the vectors of $B_1$ are unbiased with respect to $B_2$. In other words, the vectors of $B_1$ are equal superpositions of the vectors of $B_2$. In a 2-dimensional Hilbert space $\mathcal{Q}$, an example is $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$.

**Classical structures** represent the operations of copying and deletion on the classical states of a quantum observable. This corresponds to copying and deleting basis vectors.

In a Hilbert space $\mathcal{H}$ (with underlying field $K$), we can define the following maps for any orthonormal basis $\{\psi_i\}_i$:

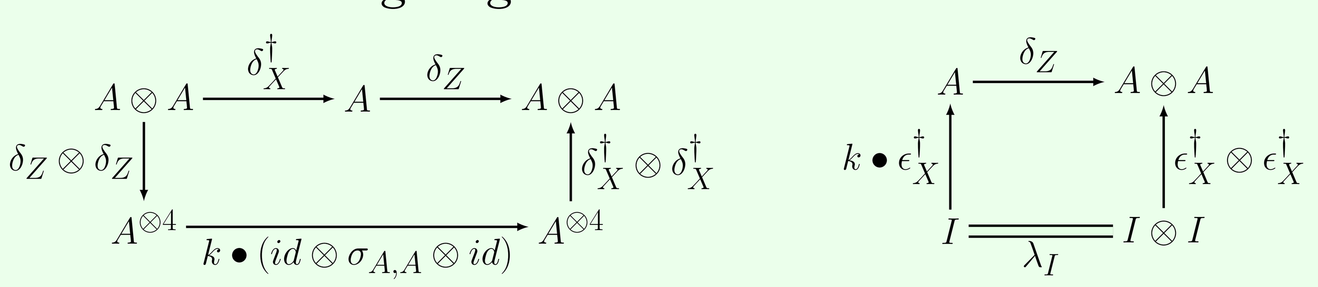
$$\delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: \psi_i \mapsto \psi_i \otimes \psi_i$$

$$\epsilon : \mathcal{H} \to K :: \psi_i \mapsto 1$$

If a pair of classical structures acts on a pair of complementary bases, we call them **complementary classical structures (CCS's)**.

## category

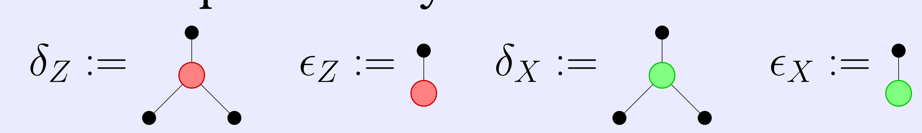
In a †-symmetric monoidal category, a classical structure $(A, \delta_Z, \epsilon_Z)$ is a cocommutative comonoid that is isometric and frobenius. These properties are sufficient to show the **spider theorem**, which states that any morphism composed of $\delta_Z$, $\epsilon_Z$, $\delta_Z^\dagger$, and $\epsilon_Z^\dagger$ is uniquely determined by the number of inputs and outputs. A pair of classical structures is complementary iff $(\delta_Z^\dagger, \epsilon_Z^\dagger, \delta_X, \epsilon_X)$ and $(\delta_X^\dagger, \epsilon_X^\dagger, \delta_Z, \epsilon_Z)$ form **scaled bialgebras** over $A$. Let $\epsilon_X^\dagger \circ \epsilon_Z = \epsilon_Z^\dagger \circ \epsilon_X$ be a scalar $k : I \to I$, then the following diagrams commute.



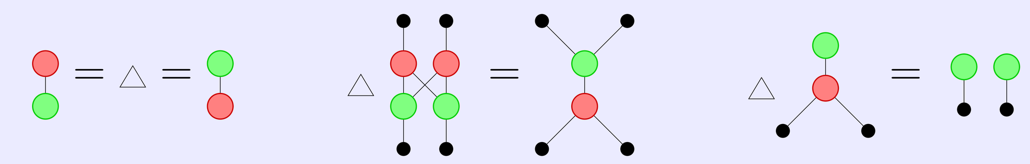

And also the above diagrams, swapping $X$ and $Z$.

## graphical

Selinger et al show that symmetric monoidal categories have natural graphical representations. In a †-SMC, we represent complementary classical structures as follows.




Using scalars $k := \triangle$ and $1/k := \triangledown$, we can express the properties of complementary classical structures as graphical equations.



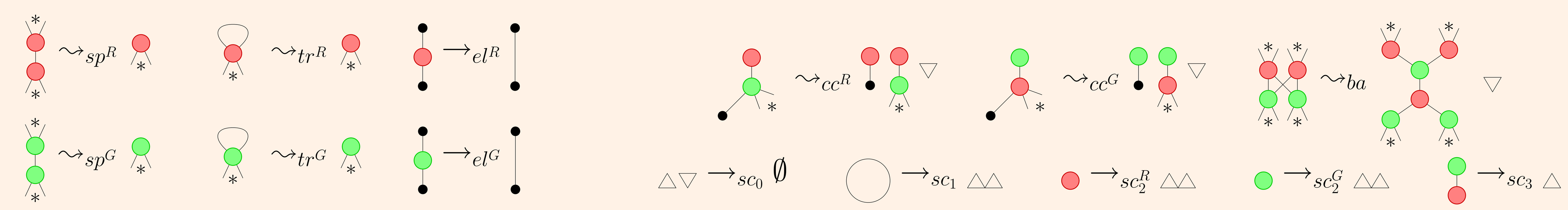
These diagrams are read top-to-bottom, where every red or green dot represents a morphism in a classical structure, and each edge represents the object $A$. We can represent tensor products as juxtaposition and arrow composition as "gluing" graphs on boundary vertices. Also, for an arrow $f : A^{\otimes m} \to A^{\otimes n}$, the graph of $f^\dagger$ is the graph of $f$ turned upside-down. Extensions of this graphical language include symbols for phase angles ( $\alpha$ ) and the Hadamard gate ( H ).

## automation

Performing graph rewrites by hand for non-trivial quantum systems can be a difficult and error-prone task. To automate this process, we present **quantomatic**. This tool provides interactive graph manipulation and rewriting. We hope to employ quantomatic's matching engine for the analysing graph rewrite systems (critical pair analysis), normalisation of quantum graphs, and quantum graph conversion (e.g. expressing graphs of CCS's as circuits or graph states).

## rewrite system

Because of the spider theorem, we can rewrite any connected subgraph of a single classical structure as an $n$-legged "spider." Therefore, we generalise the identities on classical structures to account for this property using **graph pattern rewrites rules** (denoted "$\rightsquigarrow$").




We have shown the existence of (1) a **confluent**, **terminating** extension of the above system, minus *ba* and (2) a **locally confluent** extension of the full system (that is also likely to be confluent).