Some Bad Resolutions

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> *Prove theime both by resolution and then ye shall knowe, the reason of their agremente.*
>
> Recorde, 1557

I am grateful to Alan Bundy for introducing me to many new ideas and research topics, not to mention the many folk who have been part of the *DREAM* group. By a small token of appreciation, here are some jottings around a question in automated reasoning that involves both discovery and reasoning, with of course open questions.

The starting point is about the use of Alan Robinson's resolution rule, which we can restrict here to propositional resolution. Always on the look-out for ways to restrict search, the resolver comes up with The Heuristic:

> only use the resolution rule where the size of the resolvent is not larger than either of the resolved clauses.

So the query is: does this restriction lose completeness?

The question was been asked on the newsgroup sci.logic, by Olivier Roussel, with ensuing discussion.

One piece of evidence proposed is this:

> If The Heuristic is complete, then $P = NP$.

The argument looks at propositional problems in 3-CNF, and notes that the number of distinct 3-CNF clauses that can be generated from some given axioms is polynomially bounded in the number of propositions in the given axioms. So there is a polynomial bound on the number of clauses that can be generated in a resolution proof from the given clauses using The Heuristic; if the Heuristic were complete, taking some care about the data structures used, this gives a polynomial time algorithm for 3-SAT, a classic NP-complete problem.

So, barring seismic surprises in complexity theory, The Heuristic is not complete. But can we find a concrete counter-example? One place to look is in propositional encodings of the pigeon-hole principle. Since Haken, it has been known that resolution proofs of these statements (fitting $n + 1$ pigeons into n holes) must be exponentially long, making them likely candidates. The trouble is in checking where the restriction first blocks proof – checking this ran out of time (in 1997). The case of 5 pigeons in 4 holes is reported to be a counter-example. This problem has 10 propositional variables, 6 clauses with

5 variables, and 75 clauses with 2 variables. A smaller counter-example would be better.

Here is a smallish set of clauses, in the 3-SAT class, where The Heuristic is incomplete, followed by some thoughts on how I found this.

```
{ a, b, c}           { a2,~b, c}
{ ~a, d, e }         { ~a2, d2, e2 }
{ b, c, ~d }         {~b,  c, ~d2 }
{ b, c, ~e }         {~b, c, ~e2 }

{ a1, b, ~c}         { a3,~b, ~c}
{ ~a1, d1, e1 }      { ~a3, d3, e3 }
{ b, ~c, ~d1 }       {~b, ~c, ~d3 }
{ b, ~c, ~e1 }       {~b, ~c, ~e3 }
```

Take the first set of four clauses; the first two clauses resolve to give the larger clause `{ b, c, d, e }`, and now to get a contradiction we want to get something smaller; the next two clauses will knock out `d, e` to get `{ b, c }`. Now devise variants that will result in `{ b, ~c }`, `{ ~b, c }`, `{ ~b, ~c }`; the same pattern will do, but we had better use variants that will not interact between the four sets of clauses. Of course this is a heuristic argument, but it is possible to check by machine that in fact there is no resolution proof of inconsistency without using clauses of size four (if you have a resolution system that allows this aspect of search to be controlled).

There is some symmetry reasoning involved in devising this example; Jim Molony came up with a different example, shown here:

```
{ a,  b,  c}
{ f,  g,  h}

{~a,  d,  e}                  {~f, j, ~e}
{~b,  d,  e}                  {~g, j, ~e}
{~c,  d,  e}                  {~h, j, ~e}

{~f, ~d,  i}                  {~a, ~j, k}
{~g, ~d,  i}                  {~b, ~j, k}
{~h, ~d,  i}                  {~c, ~j, k}

{~a,  e, ~i}                  {~f, ~k, ~e}
{~b,  e, ~i}                  {~g, ~k, ~e}
{~c,  e, ~i}   (* prove e *)  {~h, ~k, ~e}  (* prove ~e *)
```

This elegant solution has fewer propositional variables, though more clauses.

No smaller solutions seem to be known where this phenomenon occurs. So it is an open problem to find the smallest examples where this arises, for a chosen notion of size, and also to find a good lower bound on such examples. Simple generate and test does not look hopeful, though we might get lucky – maybe there is a systematic way to exploit the symmetry aspect of the examples above.

To place this topic in a larger context, does the following citation involving resolution reveal a seventeenth century premonition of the enterprise of "witty men" like Alan Bundy?

> *To the present impulses of sense, memory and instinct, all the sagacity of brutes may be reduced; though witty men, by analytical resolution, have chymically extracted an artificial logick out of all their actions.*
>
> Hale: "The Primitive Origination of Mankind", 1677

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